STM JOURNALS

Recent Trends in
Fluid Mechanics

ISSN: 2455-1961
Volume 10, Issue 1, 2023
DOI (Journal): 10.37591/RTFM

http://engineeringjournals.stmjournals.in/index.php/RTFM/index

Research RTFM

Composition of Ideal Fluid Flows Around Cylinder Using BilinearTransformation

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Abstract

Advent of conformal mapping eases to model a flow in sought-after domain without trade- off between laws of fluid mechanics and the theory of complex valued-functions. The far-reaching properties manifested by conformal mapping discern and facilitate solution of large class of two-dimensional flow problems. Bilinear transformation is amidst the conformal mappings. The properties, mapping a circular boundary onto horizontal line and composition of bilinear transformations is again a bilinear transformation are steppingstone to study a composition of a flow of an ideal fluid around a cylinder of circular cross-section mounted in the plane. This circumstance simplifies the proposition of combining two flows, which constitutes the primary focus of this paper. As an initial step towards advancing the investigation of fluid flow, an attempt is made herein. However, the process of adding the two fundamental flows is more intricate than it may initially appear.

Keywords: Fluid mechanics, bilinear transformation, Reynolds numbers, ideal fluid, velocity

INTRODUCTION

Flow through a circle cylinder is one of the more traditional study problems. Isaac Newton is believed to have been the first to observe the motion of tennis ball in 1671 [1]. The path of bullet is nearly parabola was the common understanding at the end of 18th century. At the beginning of 19th century in 1805, Benjamin Robins attributed the deflection in motion of bullet to the change in air resistance [2] and hence called the Robin effect. Magnus studied the motion of a fluid around spinning cylinder of uniform circular cross section in 1852 and investigated that the path deflection of cylinder is due to fluid pressure on the opposite side of the spinning cylinder, called Magnus effect [3–4]. However, the zero drag and lift force on the cylinder in terms of circulation, called Kutta-Joukoswki theorem, is presented in early 20th century by the duo, M. W. Kutta and N. Y. Zhukovsky (or Joukoswki) [5]. Rayleigh [6] observed the irregularity in flight of the tennis ball in the year 1877. Lafay [7] noted the deflection of stream lines in the vicinity of circular cylinder due to the pressure and suction in the year 1912. However, Milne-Thomson explicitly devised the complex potential for the flow in the plane when the right circular cylinder, is placed into two-dimensional flow such that its axis is perpendicular to the plane of flow, known as Milne-Thomson circle theorem [8].

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Received Date: November 09, 2023
Accepted Date: November 21, 2023
Published Date: December 06, 2023

Citation: Mohammed Mukhtar Mohammed Zabih. Composition of Ideal Fluid Flows Around Cylinder Using BilinearTransformation. Recent Trends in Fluid Mechanics. 2023; 10(1): 31–38p.

The linear partial differential equations superposition of responses is once more a solution. Irrotational flows being solution of partial differential equation $\nabla^2\phi = 0$, i.e., the Laplace equation can be superimposed to form the abounding new solutions which are complicated in nature. Sink flow, source flow, doublet flow and vortex flow are the constituents to form the combined irrotational flows. Rankine [1820–1872] took superposition of basic flows such as uniform flow and source flow to give Rankine half body flow while uniform flow and sink flow to give the flow past a Rankine oval. Flow about circular cylinder when added to free vortex generates lifting flow. But to sum up the elementary flows is not merely the mathematical operation to do. It is the composition of two flows which is not yet studied. To extend the study of fluid flow to the next level, one attempt as a first step is made here. Adding the two fundamental flows is not as simple as it seems. At first, derivative of complex conjugate of bilinear transformation is a complex representation of fluid flow around cylinder is proved. and by using the property of bilinear transformation that the composition of two bilinear transformation is again a bilinear transformation, two fluid flows around circular cylinder are composed to give the flow around same cylinder.

One-to-one conformal projection can be used to map the domain boundaries onto a line that is horizontal in the w-plane, allowing for the development of flow patterns in a specific domain in the z-plane. Bilinear transformation considered, tore off, nowadays is the mapping which can map the domain under study to the coveted one is one of its triumphs which is less appreciated and enjoyed by scientist's and engineer's. In general, a straight line is a collection of points z in the z-plane that fulfil $|z - z_1| = |z - z_2|$. Specifically, given certain restrictions, it appears geometrically possible that the set of points z lies on a real axis or horizontal line. Under certain restrictions bilinear transformation can map the circle onto a horizontal line. Taking advantage of this property we construct a flow of fluid around a cylinder in a plane.

PREREQUISITE

Theorem

If $w = \Omega(z) = \phi(x, y) + i\,\psi(x, y)$ is a one-to-one conformal mapping of the domain D in the z- plane onto a domain D' in the w-plane such that the image of the boundary C of D is a horizontal line in the w-plane, then $f(z) = \overline{\Omega'}(z)$ is a complex representation of a flow of an ideal fluid in D [9].

Milne-Thomson Circle Theorem

Let $f(z)$ be the complex velocity potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $|z| = a$. Then, on introducing the solid circular cylinder $|z| = a$ into the flow, the new complex velocity potential is given by

$w = f(z) + \bar{f}(a^2/z)$

for $|z| \geq a$. [8,10,11].

Blasius Theorem

Assuming no external forces, an incompressible fluid flows continuously and irrotationally parallel to the z-plane past a fixed cylinder, the section of which is enclosed by a closed curve ①. The action of the fluid pressure on the cylinder is equal to a force per unit length with component [X, Y] and a couple per unit length of moment M, where W is the complex potential for the flow. where

$$Y + iX = -\frac{\rho}{2}\oint \left(\frac{dw^2}{dz}\right) dz;$$

$$M = Re\left\{\left\{-\frac{\rho}{2}.\oint \overrightarrow{\ldots} z \left(\frac{dw^2}{dz}\right) dz\right\}\right. \quad (1)$$

Kutta-Joukowski Lift Theorem

$M = Re\,\{-2\oint \; z\,(\,dz\,)$

$dz\}$ [1] Think about a continuous, uniform flow that is moving past obstruction B at speed U. The flow has no singularities and there is a circulation Γ surrounding the impediment. Then, according to, the obstruction experiences a lift force L perpendicular to the flow and a drag force D parallel to the flow [10].

$D = 0$ *and* $L = -\rho U\Gamma$ [12].

D'Alembert Paradox

A body moving with constant velocity in relation to the fluid experiences zero drag force in an incompressible and inviscid potential flow. The discovery of significant drag on bodies moving in relation to fluids like air and water, particularly at high velocities that correspond with high Reynolds numbers, directly contradicts the theory of zero drag.

MAIN THEOREM

Lemma

In the complex plane the locus of z such that $|z - z_1| = |z - z_2|$ is a straight line. Moreover, it is horizontal line if $Re(z_1) = Re(z_2)$ and real axis if $\overline{z_1} = z_2$.

Theorem

If in the bilinear transformation,

$$w = \Omega(z) = \frac{az + b}{cz + d}$$

the constants a, b, c and d are such that $ad - bc \neq 0$, also $|d| = r|c|$, where $r > 0$ and

$$Re\left(\frac{a}{c}\right) = Re\left(\frac{b}{d}\right)$$

then $\overline{\Omega'}(z)$ is a complex representation of a flow of an ideal fluid in the domain $|z| > r$ in z-plane.

Proof

To prove the theorem, we check requirements of theorem (2.1) for $w = \Omega(z)$. Let D denote the domain $|z| > r$, including point at infinity in the z-plane, of the bilinear transformation $w = \Omega(z)$.

Since $|d| = r|c|$, the pole $z = -\frac{d}{c}$, only singularity, of $w = \Omega(z)$ which lies on the boundary of D. It precludes the existence of singularity inside $\overline{D}$ and ensures that $w = \Omega(z)$ is analytic there. Being bilinear transformation the mapping $w = \Omega(z)$ is one-to-one on its domain and $\Omega'(z) \neq 0$ follows from the condition that $ad - bc \neq 0$. Also, for each w in co-domain there is.

$$z = \frac{-dw + b}{cw - a} \epsilon D$$

To find the image of the boundary $|z| = r$ of D under the bilinear transformation $w = \Omega(z)$. Let us consider

$$r = |z| = |\frac{-dw+b}{cw-a}$$

$$\left|w - \frac{a}{c}\right| = w - \frac{b}{d} \qquad (1)$$

It follows from lemma ($\oint$) that equation (1) is a horizontal line in the w-plane. However, to find the image of the domain $|z| > r$ under $w = \Omega(z)$.

$$\text{let } \alpha = \frac{a}{c}, \beta = \frac{b}{d} \Longrightarrow Re\,\alpha = Re\,\beta$$

$$\Rightarrow \alpha = s + it_1 \text{ and } \beta = s + it_2,$$

Also, if $w = u + iv$ then $|z| > r \Rightarrow 2v > t_1 + t_2.$

Therefore, the exterior of the circle $|z| = r$ maps either above or below the line (1) according as $t_1 + t_2 > 0$ or $0 > t_1 + t_2.$ Thus, $w = \Omega(z)$ is one-to-one conformal mapping from $|z| > r$

onto $2v > t_1 + t_2$ of w-plane.

Corollary

If the bilinear transformations, $w = \Omega_1(z)$ and $w = \Omega_2(z)$ are two complex velocity potentials of the flow of an ideal fluid in the domain $|z| > r$ such that they differ by constant only. Then $\bar{\Omega}'_1(z)$ and $\bar{\Omega}'_2(z)$ represents same flow around the cylinder [11].

Lemma

A bilinear transformation

$$w = \Omega(z) = \frac{az + b}{cz + d}$$

maps a real axis of z-plane onto a real axis of w-plane if and only if a, b, c and d are real.

Theorem

In the bilinear transformation

$$\Omega 1(z) = \frac{az + b}{cz + d}$$

$|d| = t|c|$ where $t > 0$ and

$$\frac{a}{c} = \frac{b}{d}$$

Also, in the bilinear transformation

$$\Omega_2 \quad (z) = \frac{\alpha z + \beta}{v z + \delta}$$

α, β, γ and δ are real. If $\Omega(z) = \Omega_2(\Omega_1(z))$ then $\bar{\Omega}'(z)$ is a complex representation of a flow of an ideal fluid in the domain $|z| > t$ in z-plane.

Proof

It is evident that

$$\Omega(z) = \Omega_2(\Omega_1(z))$$

$$\Omega(z) = \frac{(\alpha a + \beta c)z + (\alpha b + \beta d)}{(\gamma a + \delta c)z + (\gamma b + \delta d)}$$

$$= \frac{pz + q}{rz + s}$$

is a bilinear transformation.

where $p = \alpha a + \beta c$, $q = \alpha b + \beta d$, $r = \gamma a + \delta c$ and $s = \gamma b + \delta d$. Also $ps - rq \neq 0$, As $\frac{a}{b}$ and $\frac{b}{d}$ are conjugates

$$\frac{a}{c} = \lambda + i\mu \Rightarrow \frac{b}{d} = \lambda - i\mu$$

$$\left|\frac{s}{r}\right| = |\frac{\gamma b + \delta d}{\gamma a + \delta c} = t$$

$$\frac{p}{r} = \frac{(\alpha\lambda + \beta) + i\alpha\mu}{(\gamma\lambda + \delta) + i\gamma\mu}$$

$$\frac{q}{s} = \frac{(\alpha\lambda + \beta) - i\alpha\mu}{(\gamma\lambda + \delta) - i\gamma\mu}$$

$$\Rightarrow \mathrm{Re}\left(\frac{p}{r}\right) = Re\left(\frac{q}{s}\right)$$

Therefore, by theorem (3.2) $\bar{\Omega}(z)$ represents a flow of an ideal fluid around a cylinder $|z| = t$ of circular cross section.

Another Proof

By virtue of lemma (3.1), since $\frac{a}{c} = \frac{b}{d}$ equation (1) shows that Ω maps a circle $|z| = t$ onto real axis which by lemma (3.4) is mapped onto real axis under Ω_2. A bilinear transformation Ω, being a composition of two bilinear transformations Ω_1 and Ω_2 is one-to-one conformal mapping of the circle $|z| = t$ onto the horizontal line $v = 0$. Therefore, by theorem (2.1), $\bar{\Omega}(z)$ represents flow of fluid in the sought-after domain.

Theorem

If the bilinear transformation $w = \Omega$ (z) is complex velocity potential for the steady flow of an ideal fluid around circular cylinder whose boundary in the plane is ✓, then the net force exerted by the fluid on cylinder is zero. Further the moment of couple is zero.

Proof

Suppose a complex velocity potential be given by

$$w = \frac{az + b}{cz + d}$$

The net force, by Blasius theorem is

$$Y + iX = -\frac{\rho}{2}\oint \left(\frac{dw^2}{dz}\right) dz$$

$$= \frac{-\rho}{2}\oint \frac{(ad - bc)^2}{(cz + d)^4} dz$$

$= -\frac{\rho}{2}(ad - bc)^2 \times 2\pi\, i \times$ (Residue of the function

$\frac{1}{(cz+d)^4}$ at $z = -\frac{d}{c}$=0

Also, moment about O is

$$M = Re\left\{-\frac{\rho}{2}\oint z\left(\frac{dw^2}{dz}\right) dz\right\}$$

$$= Re\left\{-\frac{\rho}{2}(ad - bc)^2 \oint \frac{z}{(cz+d)^4} dz\right\}$$

$= Re\left\{-\frac{\rho}{2}(ad - bc)^2 \times 2\pi\, i \times \oint \text{Residue of the function } \frac{z}{(cz+d)^2} \text{ at } z = -\frac{d}{c}\right\}$=0

EXAMPLES

Example

Construct a flow of an ideal fluid in the domain D; $|z| > 1$ in the plane. Consider the mapping be defined on D by

$$w = \Omega(z) = \frac{iz + 1}{z + i}$$

$$u(x, y) = \frac{2x}{x^2+(1+y)^2}$$

$$v(x, y) = \frac{x^2 + y^2 - 1}{x^2 + (1 + y)^2}$$

clearly $w = \Omega$ (z) being bilinear transformation is one-to-one conformal mapping of D. For each w there is $z = \frac{1-iw}{w-i} \epsilon D$ and hence onto mapping.

To find image of the domain D and its boundary, take $|z| = 1 \Rightarrow v = 0$. If $|z| > 1 \Rightarrow v > 0$

Therefore, the domain D and its boundary are mapped onto upper half plane, $Im\ w > 0$ and real axis, $v = 0$ (i.e. horizontal line) of the w-plane respectively.

The stream lines of the flow are

$$\frac{x^2+y^2-1}{x^2+(x+y)^2} = constant$$

$$f(z) = \Omega(z) = \frac{i-1}{(z)^2}$$

is a complex representation of fluid flow.

Example

Construct a flow of an ideal fluid in the domain D; $|z| > 1$ in the plane.

Construction 1

Consider the mapping

$$w = \Omega 1(z) = (1 + i)\frac{z+i}{z-1}$$

$$u(x, y) = \frac{x^2 + y^2 - 2x + 2y + 1}{(x-1)^2 + y^2}$$

$$v(x, y) = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2}$$

If w is any point in co-domain then $z = \frac{w-1+i}{w-i-1} \epsilon D$, hence onto mapping. Same argument as in above Section of Example , gives the stream lines of the flow

$$\frac{x^2 + y^2 - 1}{(x-1)^2 + y^2} = constant.$$

$$f(z) = \Omega(z) = \frac{2i}{(z-1)^2}$$

Construction 2

Consider the mapping

$$w = \Omega 2(z) = i\ \frac{z+1}{z-1}$$

$$u(x, y) = \frac{2y}{(x-1)^2 + y^2}$$

$$v(x, y) = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2}$$

It can easily be verified the required condition to give stream lines

$$\frac{x^2 + y^2 - 1}{(x-1)^2 + y^2} = constant.$$

$$f(z) = \Omega_2(z) = \frac{2i}{(z-1)^2}$$

It should be noted that the bilinear transformations in construction 1 and construction 2 of example (4.2) are such that

$\Omega_1(z) - \Omega_2(z) = 1$

it implies that

$\Omega_1(z) = \Omega_1(z)$

and hence the flows are same, which is what the corollary (3.3) says. Talking in another language, distinct bilinear transformations could be the complex velocity potential for the same flow in a given domain.

Finally, we will find composition of two flows.

Example

If θ is a real constant and

$$w = \Omega_1(z) = \frac{ie^{\theta}z - i}{z - e^{\theta}}$$

$$w = \Omega_2(z) = \frac{z+2}{3z+4}$$

Also, $\Omega(z) = \Omega_2(\Omega_1(z))$ then show that $\bar{\Omega'}(z)$ is a complex representation of a flow of an ideal fluid around a cylinder of circular cross section centered at ($cosh\theta$, 0) and radius $sinh\theta$.

Let $z = x + iy$, $w = u + iv$. Consider the circle centered at ($cosh\theta$, 0) and radius $sinh\theta$.

The equation of a circle is

$$x^2 + y^2 - 2x\,cosh\theta + 1 = 0$$

$$\frac{i(ze^{\theta} - 1)}{z - e^{\theta}} + \frac{i(ze^{\theta} - 1}{z - e^{e^{\theta}}} = 0$$

$$w - \bar{w} = 0$$

$$v = 0$$

The image of real axis $y = 0$ under $w = \Omega_2(z)$ is

$$u + iv = \frac{x+2}{3x+4} \Rightarrow v = 0$$

The real axis in the co-domain of $w = \Omega_2(z)$.

Now

$\Omega(z) = \Omega_2(\Omega_1(z))$

$$= \frac{(ie^{\theta} + 2)z - (i + 2e^{\theta})}{(3ie^{\theta} + 4)z - (3i + 4e^{\theta})}$$

maps the circle centered at ($cosh\theta$, 0) and radius $sinh\theta$ onto the horizontal line viz. real axis in the co-domain of Ω_2. Thus, by the theorem (3) $\bar{\Omega'}(z)$ gives required complex representation of fluid flow [12].

CONCLUSION

Property of bilinear transformation- mapping the circle onto line and horizontal line subject to some constraints eases to construct the ideal fluid flow around a circular cylinder (theorem 3.) i.e. to derive the complex velocity potential of flow of an ideal fluid commensurable with Milne- Thomson circle theorem (2.2). If two bilinear transformations satisfying the conditions of theorem 4., then the conjugate of derivative of the composite map of these two mapping represents the complex form of an ideal fluid flow. This paves the way to engender the new flows from the known flows by taking their composition. However, this method is limited to the flow of an ideal fluid past a circular cylinder only. But the little extension can open an avenue to produce new flows as a composition of existing flows in general.

Net force exerted by fluid on a cylinder and moment of couple is evaluated using theorem due to Blasius (2.3) both of which are zero. Ideally, we expect in practice that the fluid should experience a nonzero force on cylinder. This discrepancy between experimental reality and theoretical predictions is due to the negligence of viscous effect, which is exactly what the D'Alembert's paradox (2.5) says for the moving body relative to fluid motion. Thus, the D'Alembert's paradox is verified for the flow derived by using bilinear transformation. Force and moment (zero) derived in this chapter are consistent with the well-known theorems due to Kutta- Joukowski (2.4).

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